Momentum of Electromagnetic field: The force on a region containing both charges and current is

$$F = \int_V (\rho E + J \times B)\, d\tau \quad -(11)$$

If P_{mech} → sum of momenta of all the particles

$$\frac{dP_{mech}}{dt} = \int_V (\rho E + J \times B)\, d\tau \quad -(12)$$

From Maxwell's eq^n's

$$\rho = \nabla \cdot D \ ; \quad J = \nabla \times H - \frac{\partial D}{\partial t}$$

$$\Rightarrow \frac{dP_{mech}}{dt} = \int_V \left\{ (\nabla \cdot D) E + \left(\nabla \times H - \frac{\partial D}{\partial t}\right) \times B \right\} d\tau$$

$$= \int_V \left\{ (\nabla \cdot D) E + B \times \frac{\partial D}{\partial t} - B \times (\nabla \times H) \right\} d\tau$$

Since $\frac{\partial}{\partial t}(D \times B) = D \times \frac{\partial B}{\partial t} + \frac{\partial D}{\partial t} \times B$

$$\frac{dP_{mech}}{dt} = \int_V \left[(\nabla \cdot D) E + \left(D \times \frac{\partial B}{\partial t}\right) - \frac{\partial}{\partial t}(D \times B) - B \times (\nabla \times H) \right] d\tau$$

Because $\nabla \cdot B = 0$, addition of $(\nabla \cdot B) H$ to the square bracket does not alter the result.

Therefore

$$\frac{dP_{mech}}{dt} + \frac{d}{dt} \int_V (D \times B)\, d\tau = \int_V \left[(\nabla \cdot D) E + (\nabla \cdot B) H - \{ D \times (\nabla \times E) \} - \{ B \times (\nabla \times H) \} \right] d\tau \quad -(13)$$

$\left(\because \nabla \times E = -\frac{\partial B}{\partial t}\right)$

The integral in the second term of left-hand side represents momentum. Since it is not associated with mass of the particles.

It consists only of fields, → electromagnetic moment of field. R.H.S → can be converted in surface integral → momentum flow.
vector $g = [D \times B]$ → electromagnetic field density

$$g = [D \times B] = [\varepsilon E \times \mu H] = \mu\varepsilon [E \times H]$$

g ↓ momentum density vector

$$= \mu \varepsilon N \qquad \text{--- (14)}$$

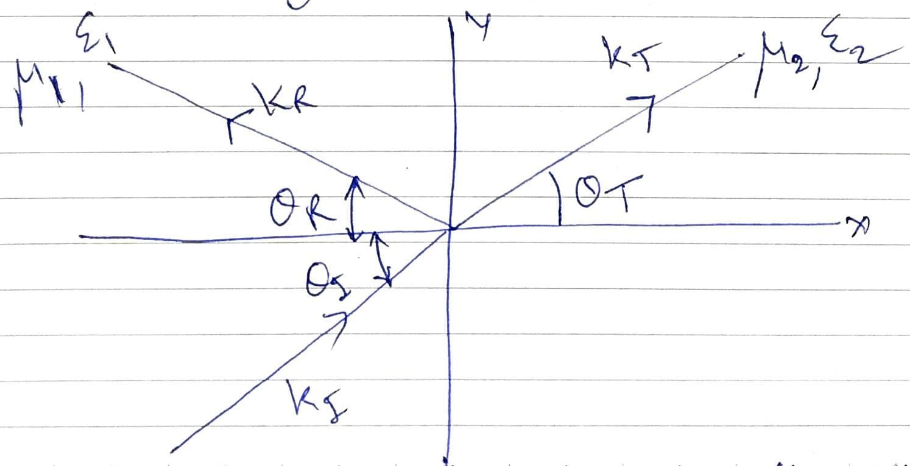
N ↳ Poynting vector

Electromagnetic waves in Bounded Medium

Behaviour of e.m. waves at the boundaries between different medium. We shall limit our discussion to plane boundaries only.

Reflection and Refraction of Plane wave at a plane Interface

Consider two non-conducting ($\sigma = 0$) dielectric media referred to as '1' and '2' characterized by constants μ_1, ε_1 and μ_2, ε_2 and separated by a plane boundary - the plane $x = 0$



Suppose a plane e.m. wave is incident obliquely on the plane boundary.

There will be in general both reflected wave and a transmitted wave

We can express the fields for the incident, reflected and transmitted waves as;

$$E_I = E_{0I} \exp\{i(k_I \cdot r - \omega_I t)\}, \quad H_I = \frac{k_I \times E_I}{\omega_I \mu_1} \quad -(1)$$

$$E_R = E_{0R} \exp\{i(k_R \cdot r - \omega_R t)\}, \quad H_R = \frac{k_R \times E_R}{\omega_R \mu_1} \quad -(2)$$

and

$$E_T = E_{0T} \exp\{i(k_T \cdot r - \omega_T t)\}, \quad H_T = \frac{k_T \times E_T}{\omega_T \mu_2} \quad -(3)$$

$I, R, T \rightarrow$ represent incident, reflected and transmitted waves respectively.

$E_{I0}, E_{0R}, E_{0T} \rightarrow$ Time-independent scalar amplitudes (may be complex)

The tangential components of E and H can be continuous across the boundary at all points and all times only if the exponentials are the same at the boundary for all three fields. This is possible if

$$\omega_I = \omega_R = \omega_T$$

i.e., the frequency is unchanged in the reflected and transmitted waves, and

$$k_I \cdot r = k_R \cdot r = k_T \cdot r \quad -(4)$$

This shows that all the propagation vectors are coplanar. If we choose r to lie in the boundary plane (i.e. $\hat{e}_n \cdot r = 0$) ($\hat{e}_n \rightarrow$ unit vector ~~normal to~~ the plane) and in the plane of propagation vector, it follows that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \quad \text{---(5)}$$

Propagation vectors k_I and k_R → are in the same medium hence are equal in medium.

Therefore

$$\theta_I = \theta_R \quad \text{---(6)}$$

Since $k_I \sin\theta_I = k_T \sin\theta_T$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} = \sqrt{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}} \quad \left(\because k = \omega\sqrt{\varepsilon\mu}\right)$$

For non-magnetic materials, we may assume $\mu_2 = \mu_1$.

Therefore

$$\frac{\sin\theta_I}{\sin\theta_T} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \quad \text{Snell's law} \quad \text{---(7)}$$

n_1, n_2 → refractive indices of the media '1' and '2' respectively.

eqn (6), (7), → simpler laws of geometrical optics → we are already familiar.

<u>Relationship between the various field vectors</u>

Divergence eqn's $\nabla \cdot D = \rho$ and $\nabla \cdot B = 0$ can be obtained by applying the divergence operator to the remaining Maxwell's eqn's involving E and H, the boundary conditions on D_n and B_n are automatically satisfied provided the conditions on $\vec{E}_t$ and H_t are met.

The conditions are:

$$(E_I + E_R) \times \hat{e}_n = E_T \times \hat{e}_n \quad \text{---(8)}$$

and

$$(H_I + H_R) \times \hat{e}_n = H_T \times \hat{e}_n \quad \text{---(9)}$$